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THE TORSION OF CYLINDRICALLY ANISOTROPIC TWO-DIMENSIONALLY INHOMOGENEOUS SOLIDS OF REVOLUTION†

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The form of the anisotropy for which an accurate solution is constructed for the stress and displacement of functions in the form of an infinite integral operator is established for the shear modulus, represented in the form of the product of continuous functions of two cylindrical coordinates of an orthotropic cylinder. This operator contains an arbitrary function of the complex variable, in addition to well-known functions. For power and exponential-power inhomogeneities, the case in which the operator degenerates into a monomial is distinguished, and, using this as an example, the problem is solved for a hollow cylinder with mixed boundary conditions.

1. INITIAL EQUATIONS

Pure torsion of an orthotropic solid of revolution, the anisotropy axis of which coincides with the geometrical axis of symmetry of the solid, in cylindrical coordinates r, θ, z, is represented by a system of partial differential equations of elliptic type with variable coefficients [1, 2]

$$\frac{\partial\psi}{\partial r}-P(r,z)\frac{\partial\varphi}{\partial z}=0, \quad \frac{\partial\psi}{\partial z}+Q(r,z)\frac{\partial\varphi}{\partial r}=0 \tag{1.1}$$

$$P=r^3G_1(r,z), \quad Q=r^3G_2(r,z)$$

Here ψ is the stress function, φ is the displacement function, and $G_{\theta z}=G_1(r,z)$, $G_{r\theta}=G_2(r,z)$ are the shear moduli.

The components of the stresses $\tau_{\theta z}=\tau_1(r,z)$, $\tau_{r\theta}=\tau_2(r,z)$, the displacement $u_\theta=u(r,z)$ and the resultant torque M at the ends of the solid are defined by the equations

$$\tau_1=\frac{1}{r^2}\frac{\partial\psi}{\partial r}=rG_1\frac{\partial\varphi}{\partial z}, \quad \tau_2=-\frac{1}{r^2}\frac{\partial\psi}{\partial z}=rG_2\frac{\partial\varphi}{\partial r} \tag{1.2}$$

$$u_\theta=r\varphi, \quad M=2\pi\int_0^R r^2\tau_1 dr=2\pi[\psi(R,z)-\psi(0,z)]$$

When $z=0$ and $z=H$ (where R and H are the radius and length of the circular cylinder) we obtain the conditions at the ends.

When the shear modulus varies as a function of the radius only, solutions have been constructed using the method of separation of variables [1, 2] and another method [3] for particular forms of the shear modulus, and general solutions [4, 5] in the form of integral and differential complex series for arbitrarily specified moduli of the radius $G_1(r)$ and $G_2(r)$. A self-similar solution for the stress function $\psi=\psi(r/z)$ was obtained in [2] for a circular isotropically inhomogeneous cone for particular forms of the moduli $G_2=cG_1$ ($c=\text{const}$), which depend on the two coordinates r and z.

We will construct new solutions for the torsion of a circular cylinder when the shear moduli can be represented in the form of functions of two coordinates

$$G_1=c_1p_1(\xi)q_1(\eta), \quad G_2=c_2p_2(\xi)q_2(\eta) \tag{1.3}$$

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where c_1 and c_2 are arbitrary dimensional constants, and p_1, p_2, q_1 and q_2 are arbitrary continuous functions of the arguments

$$\xi = r/R, \quad \eta = (z+H)/H \tag{1.4}$$

We will now write the system of equations (1.1) in the form

$$H\frac{\partial\psi}{\partial\xi} = c_1 R^4 \xi^3 p_1 q_1 \frac{\partial\varphi}{\partial\eta}, \quad \frac{\partial\psi}{\partial\eta} = -c_2 R^2 H \xi^3 p_2 q_2 \frac{\partial\varphi}{\partial\xi} \tag{1.5}$$

Eliminating φ and ψ, we obtain two dimensionless second-order differential equations equivalent to system (1.5), which henceforth will be fundamental

$$\xi^3 p_2 \frac{\partial}{\partial\xi}\left(N\frac{\partial\psi}{\partial\xi}\right) + a^2 q_1 \frac{\partial}{\partial\eta}\left(q_2^{-1}\frac{\partial\psi}{\partial\eta}\right) = 0 \tag{1.6}$$

$$q_2 \frac{\partial}{\partial\xi}\left(\xi^3 P_2 \frac{\partial\varphi}{\partial\xi}\right) + a^2 \xi^3 p_1 \frac{\partial}{\partial\eta}\left(q_1\frac{\partial\varphi}{\partial\eta}\right) = 0 \tag{1.7}$$

Here

$$N = \xi^{-3} p_1^{-1}, \quad a^2 = c_1 R^2/(c_2 H^2) \tag{1.8}$$

(a is a dimensionless constant).

2. METHOD OF SOLUTION

We will take the equation for the stresses (1.6) as the initial equation. By analogy with the construction of the solution for a second-order ordinary differential equation with variable coefficients in the form of arbitrary series, each term of which is defined in terms of the previous term and, in the final analysis, all the terms of the series are expressed by corresponding quadratures, which depend on the coefficients of the initial equation and the initial conditions [6, p. 261]. We will seek the function ψ in the form

$$\psi = \Sigma\alpha_k(\xi)\beta_k(\eta)w_k(\zeta) \tag{2.1}$$

$$\zeta = u + iv, \quad u = \int \rho_1(\xi)d\xi, \quad v = \int \rho_2(\eta)d\eta \tag{2.2}$$

The summation in (2.1) is carried out from $k = 0$ to $k = \infty$, $w_k(\zeta)$ are arbitrary analytic functions of the complex variables ζ, and α_k, β_k, ρ_1, ρ_2 are the simplest (particular) values of the functions for which Eq. (1.6) is satisfied.

We will introduce the corresponding derivatives of (2.1) and (2.2) into Eq. (1.6) and group terms. As a result we obtain an equation which we will write as follows:

$$\begin{aligned}&\Sigma\left[\xi^3 p_2 \beta_k\left(N\alpha_k'' + \frac{dN}{d\xi}\alpha_k'\right) + a^2 q_1 q_2^{-2}\alpha_k\Delta_1(\eta)\right]w_k + \\ &+\Sigma\left\{\xi^3 p_2 \beta_k\left[2\rho_1 N\alpha_k' + \left(\rho_1\frac{dN}{d\xi} + N\rho_1'\right)\alpha_k\right] + a^2 q_1 q_2^{-2}\alpha_k\Delta_2(\eta)\right\}w_k' + \Delta_0(\xi,\eta)\Sigma\alpha_k\beta_k w_k'' = 0\end{aligned} \tag{2.3}$$

$$\begin{aligned}&\Delta_1(\eta) = q_2\beta_k'' - q_2'\beta_k', \quad \Delta_0 = \frac{p_2}{p_1}\rho_1^2 - \frac{a^2 q_1}{q_2}\rho_2^2 \\ &\Delta_2 = i[2\rho_2 q_2 \beta_k' + (q_2\rho_2' - \rho_2 q_2')\beta_k]\end{aligned} \tag{2.4}$$

The function $\Delta_0(\xi, \eta)$ is independent of the summation sign, $\Delta_1(\eta)$ is a real function and $\Delta_2(\eta)$ is a pure imaginary expression.

Assuming $\Delta_0 = 0$, we express ρ_1 and ρ_2 in the form

$$\rho_1^2 = p_1/p_2, \quad \rho_2^2 = q_2/(a^2 q_1) \tag{2.5}$$

From the equations $\Delta_1 = 0$ and $\Delta_2 = 0$, after integration and using expressions (2.5), we obtain the following connecting relations for β_k

$$\beta_k = A_k \int q_2 d\eta = B_k (q_1 q_2)^{1/4} \quad (k = 0, 1, 2 \ldots) \tag{2.6}$$

Equations (2.6) will be equivalent, for example, in the following cases.

1. A power relationship

$$q_1 = \eta^{\nu}, \quad q_2 = \eta^{\mu} \quad (\nu = 3\mu + 4) \tag{2.7}$$

(ν and μ are rational numbers).

In (2.6), by a suitable choice of the constants of integration A_k and B_k we can always arrange for the following equalities to be satisfied

$$\gamma A_k = B_k = 1 \quad (k = 0, 1, 2 \ldots)$$

(γ is a constant which appears as a result of evaluating the integral). Here it is important that the function β_k becomes independent of the summation index.

In this case we obtain

$$\beta_k = \beta = \eta^{\mu+1}, \quad A_k/(\mu + 1) = B_k = 1 \quad (k = 0, 1, 2 \ldots) \tag{2.8}$$

2. Power functions

$$q_1 = b^{\nu\eta}, \quad q_2 = b^{\mu\eta}, \quad \beta = b^{\mu\eta} \quad (\nu = 3\mu) \tag{2.9}$$

(b is an arbitrary positive number including the exponent).

This list can be continued; for example, for the functions $q_1 = \eta \ln^{\nu}\eta$, $q_2 = \eta^{-1} \ln^{\mu}\eta$, $\nu = 3\mu + 4$ we obtain $\beta = \ln^{\mu+1}\eta$.

The functions (2.8) and (2.9) can be used to confirm the corresponding experimental relationships.

Taking into account the fact that all Δ_i are equal to zero in (2.4), and $\beta_k = \beta$ in (2.8) and (2.9) is independent of the index of summation, we can take the quantity $\xi^3 p_2 \beta$ outside the summation sign and shorten it, and impose the following conditions on the remaining terms

$$2\rho_1 N \alpha_0' + \left(\rho_1 \frac{dN}{d\xi} + N\rho_1' \right) \alpha_0 = 0 \tag{2.10}$$

$$2\rho_1 N \alpha_k' + \left(\rho_1 \frac{dN}{d\xi} + N\rho_1' \right) \alpha_k = \frac{dN}{d\xi} \alpha_{k-1}' + N\alpha_{k-1}'' \tag{2.11}$$

$$w_k' = -w_{k-1} \quad (k = 1, 2, \ldots) \tag{2.12}$$

For these equalities, Eq. (2.3) is satisfied identically.

It follows from (2.12) that

$$w_k = (-1)^k \underbrace{\iint \ldots \int}_{k} w(\xi) \underbrace{d\zeta d\zeta \ldots d\zeta}_{k} \tag{2.13}$$

Here $w(\zeta) = w_0(\zeta)$ is an arbitrary analytic function of the complex variable ζ defined in (2.2).

We obtain from Eq. (2.10)

$$\alpha_0 = \frac{A_0}{a} \xi^{3/2} (p_1 p_2)^{1/4} \quad \left(\frac{A_0}{a} = 1 \right) \tag{2.14}$$

Note that the homogeneous part of Eq. (2.11) has the same form as Eq. (2.10), while the right-hand side is a well-known function, defined along the "chain" from the preceding to the next equation. In the final analysis this homogeneous part is predetermined by the function (2.14).

Using the method of variation of the constant of integration in the solution for the homogeneous part of the equation, we obtain a particular solution of the inhomogeneous equation (2.11), which can be written in the form

$$\alpha_k = e^{-\int Q d\xi} \int R_{k-1}(\xi) e^{\int Q d\xi} d\xi \tag{2.15}$$

where

$$Q = \frac{1}{2}\frac{d \ln N\rho_1}{d\xi}, \quad R_{k-1} = \frac{1}{2\rho_1}\left(\frac{d\ln N}{d\xi}\alpha'_{k-1} + \alpha''_{k-1}\right) \quad (k = 1,2\ldots) \tag{2.16}$$

Knowing (2.15) and β in (2.8) and (2.9), taking (2.13) into account, we will write the series (2.1) in the form of a sign-varying integral operator of the type [7, 8]

$$\psi = \beta\Sigma(-1)^k \alpha_k \int w(\zeta) d\zeta^k \tag{2.17}$$

Here we have introduced the conventional form of writing the k-tuple integral (2.13). When $k = 0$ we have $\int w d\zeta^0 = w(\zeta)$.

The real and imaginary parts of (2.17) and their linear combination are real solutions of Eq. (1.6). Specifying the function $w(\zeta)$ arbitrarily we obtain inverse boundary-value problems, some of which turn out to be suitable in practice. When solving direct boundary-value problems we take the function w in the form of a converging exponential series

$$w = \Sigma A_n e^{n\omega\zeta} + B_n e^{-n\omega\zeta} \tag{2.18}$$

in which ω, A_n, B_n are arbitrary real (in general, complex) constants, determined from the boundary conditions of the specific problem.

Here the summation, as always henceforth, is carried out from $n = 1$ to $n = \infty$.

Note that the functions (2.1) can also be specified in differential form, similar to that in [8], if in Eq. (2.3), together with $\Delta_i = 0$ (2.4), instead of (2.10)–(2.12), we impose conditions of the form

$$\begin{aligned} &N\alpha''_0 + N'\alpha'_0 = 0, \quad \alpha_0 = \int \xi^3 p_2(\xi) d\xi \\ &N\alpha''_k + N'\alpha'_k = 2\rho_1 N\alpha'_{k-1} + (\rho_1 N' + N\rho'_1)\alpha_{k-1} \\ &w_k = -w'_{k-1}, \quad (k = 1,2\ldots) \end{aligned} \tag{2.19}$$

which we will not consider in detail.

3. A POWER INHOMOGENEITY

We will consider, as an example, the case of a power relationship

$$p_1 = \xi^p, \quad p_2 = \xi^q \tag{3.1}$$

where p and q are rational numbers.

In this case, from (2.14)–(2.15) we obtain the equations

$$\alpha_0 = \xi^m, \quad Q = -m\xi^{-1} \quad \left(m = \frac{p+q}{4} + \frac{3}{2}\right) \tag{3.2}$$

$$R_{k-1} = \frac{1}{2}\xi^{(q-p)/2}[\alpha''_{k-1} - (p+3)\xi^{-1}\alpha'_{k-1}] \tag{3.3}$$

$$\alpha_k = \xi^m \int \xi^{-m} R_{k-1}(\xi) d\xi \tag{3.4}$$

For $k = 1$ we obtain

$$R_0 = A\xi^n, \quad \alpha_1 = B\xi^t, \quad A = \frac{m}{2}(m-p-4)$$
$$B = A/(n-m+1), \quad n = \frac{1}{4}(3q-p-2), \quad t = n+1-m \tag{3.5}$$

For $k = 2$ we will have

$$\alpha_2 = D\xi^s, \quad s = \frac{1}{4}(5q-3p+2), \quad D = \frac{B(m+1)(n-p-3)}{2(n-m)+q-p} \tag{3.6}$$

This process can be continued and a recurrence relation can be established for α_k.

It follows from (3.5) and (3.6) that if $A = 0$, i.e. $m = 0$ ($q = -p - 6$) or $m = p + 4$ ($q = 3p + 10$), series (2.17) is discontinued at the first term. If $B \neq 0, m = -1$ or $n = p + 3$ ($q = (5p + 14)/3$), it becomes a two-term expression and so on. In each individual case the question of the convergence of series (2.17) does not arise.

We will consider in more detail the imaginary part of (2.18) and the case when $m = p + 4$, while the value of β is determined from one of the formulae (2.8) and (2.9).

In this case, for β in the form (2.8), the moduli (1.4) and the function ψ (2.17) take the form

$$G_1 = c_1\xi^p\eta^{3\mu+4}, \quad G_2 = c_2\xi^{3p+10}\eta^\mu \tag{3.7}$$

$$\psi = \xi^{p+4}\eta^{\mu+1}\sum(A_n e^{n\omega u} - B_n e^{-n\omega u})\sin n\omega\upsilon \tag{3.8}$$

$$u = -\frac{1}{p+4}\xi^{-(p+4)}, \quad \upsilon = -\frac{1}{\mu+1}\eta^{-(\mu+1)}$$

Introducing the function (3.8) into the system of equations (1.5) we obtain, in the usual way, the displacement function

$$\varphi = \frac{p+4}{c_2R^2H}\sum\frac{1}{(n\omega)^2}[A_n e^{n\omega u}(n\omega u - 1) +$$
$$+B_n e^{-n\omega u}(n\omega u + 1)]\left[(\mu+1)\sin n\omega\upsilon + \frac{n\omega}{a}\eta^{\mu+1}\cos n\omega\upsilon\right] \tag{3.9}$$

For the case when the quantity β is defined by (2.9), i.e. for the modulus (1.3)

$$G_1 = c_1\xi^p e^{3\mu\eta}, \quad G_2 = c_2\xi^{3p+10}e^{\mu\eta} \tag{3.10}$$

we obtain that the functions ψ and φ in (3.8) and (3.9) will have the same form as for (2.9) with the sole difference that instead of the function $\eta^{\mu+1}$ we must introduce the expression for $e^{\mu\eta}$ and replace the factor $(\mu + 1)$ by μ with $\sin n\omega\upsilon$. The variable u then remains unchanged while $\upsilon = -(a\mu)^{-1}e^{-\mu\eta}$.

Note that, in the first case (2.8) for $u = -(\upsilon = a^{-1}\ln\eta)$ and in the second case (2.9) for $\mu = 0$ ($\upsilon = \eta/a$), formulae (3.8) and (3.9) are simplified and are identical with one another (taking into account, of course, the different expressions for the variable υ), namely

$$\psi = \xi^{p+4}\sum(A_n e^{n\omega u} - B_n e^{-n\omega u})\sin n\omega\upsilon$$

$$\varphi = \frac{p+4}{R^3\sqrt{c_1c_2}}\sum\frac{1}{n\omega}[A_n e^{n\omega u}(n\omega u - 1) + B_n e^{-n\omega u}(n\omega u + 1)]\cos n\omega\upsilon \tag{3.11}$$

(the value of the constant a in (1.8) is taken into account). Here, for the case (2.8) we must take $\upsilon = a^{-1}\ln\eta$, while for the case (2.9) we must take $\upsilon = \eta/a$.

By substituting (3.11) into (1.5)–(1.7) it can be shown that they are satisfied for cases (3.7) and (3.10).

To abbreviate the notation we will confine ourselves to (3.11) and (2.8), and we will write for these the stresses (1.2), determined from one of the formulae (3.11)

$$\tau_1 = -\frac{p+4}{R^3}\xi^{p+1}\sum[A_n e^{n\omega u}(n\omega u-1)+B_n e^{-n\omega u}(n\omega u+1)]\sin n\omega\upsilon \tag{3.12}$$

$$\tau_2 = -\sqrt{\frac{c_2}{c_1}}\eta^{-1}\xi^{p+2}\sum n\omega(A_n e^{n\omega u} - B_n e^{-n\omega u})\cos n\omega\upsilon$$

Correspondingly, the displacement u_θ (1.2) is also defined in terms of (3.11).

4. A BOUNDARY-VALUE PROBLEM

Using the example of (3.11) we consider a problem with mixed boundary conditions for a coaxial hollow circular cylinder of radii R and R_1 ($R > R_1$), the ends of which are free from forces under the following conditions.

When $z = 0$ ($\eta = 1$), $z = H$($\eta = 2$)

$$\tau_1 = 0, \quad \left.\frac{\partial\psi}{\partial\xi}\right|_{\eta=1,\eta=2} = 0 \tag{4.1}$$

When $r = R$($\xi = 1$)

$$\tau_2 = -R^{-2}H^{-1}\xi^{-2}\left.\frac{\partial\psi}{\partial\eta}\right|_{\xi=1} = f_1(\eta)$$

$$r = R_1\left(\xi_1 = \frac{R_1}{R_2} < 1\right): \quad u_\theta = R\xi\varphi|_{\xi=\xi_1} = f_2(\eta) \tag{4.2}$$

Here f_1 and f_2 are specified piecewise-continuous functions, bounded in the interval $\eta \in (1.2)$.

If we put $\omega = a\pi/\ln 2$ for the case (2.8) and take $\omega = 2a\pi$ for the case (2.9), conditions (4.1) for the functions ψ (3.11) are satisfied. To satisfy conditions (4.2) we expand the functions f_1 and f_2 in Fourier series in $\cos n\omega\upsilon$ in the interval $\upsilon \in (0, \ln 2)$, and using Fourier's method, we obtain the equalities

$$n\omega(A_n e^c - B_n e^{-c}) = D_n, \quad h_1 A_n e^{n\omega u_1} + h_2 B_n e^{-n\omega u_1} = n\omega E_n$$

from which we obtain the formulae

$$A_n = \Delta_1/\Delta, \quad B_n = \Delta_2/\Delta, \quad \Delta_1 = h_2 e^{-n\omega u_1}D_1 + (n\omega)^2 e^{-c}E_n$$
$$\Delta_2 = (n\omega)^2 e^c E_n - h_1 e^{n\omega u_1}D_n, \quad \Delta = (n\omega)(h_2 e^{c-n\omega u_1} + h_1 e^{n\omega u_1 - c}) \neq 0 \tag{4.3}$$

Here D_n and E_n are the known coefficients of the expansion of the functions $\Phi_1(\eta)$ and $\Phi_2(\eta)$ in Fourier series

$$E_n, D_n = \frac{2}{b}\int_0^b \Phi_i(e^{a\upsilon})\cos n\omega\upsilon\, d\upsilon \quad (i = 1,2) \tag{4.4}$$

and we have introduced the notation

$$c = -\frac{n\omega}{p+4}, \quad b = \frac{\ln 2}{a}, \quad h_1 = n\omega u_1 - 1, \quad h_2 = n\omega u_1 + 1$$

$$u_1 = -\frac{\xi^{-(p+4)}}{p+4}, \quad \Phi_1 = -R^2 H\eta f_1(\eta), \quad \Phi_2 = \frac{R^2\sqrt{c_1 c_2}}{(p+4)H\xi_1}f_2(\eta) \tag{4.5}$$

The problem is simplified for a continuous cylinder. In this case, depending on what is specified on the surface (the stress or the displacement), we use one of the functions (3.11), in which we must put the constant A_n or B_n equal to zero as being redundant.

Note that it follows from (3.12) and $p > 0$ for problem (4.1) and (4.2), that, compared with the internal section ($\xi < 1$), the largest stresses occur at the point of the external contour ($\xi = 1$) and τ_2 decreases in modulus from the end $z = 0$ ($\eta = 1$) to the end $z = H$ ($\eta = 2$). For a thin section ($R = R_1$, $\xi_1 \approx 1$) the stresses are approximately equal at points of the internal and external contours. This agrees with the conclusions for the homogeneous case. Similar conclusions apply to the displacements, which follows from (1.2) and (3.11).

REFERENCES

1. LOMAKIN V. A., *Theory of Elasticity of Non-uniform Bodies*. Izd. MGU, Moscow, 1976.
2. LEKHNITSKII S. G., *Torsion of Anisotropic and Non-uniform Rods*. Nauka, Moscow, 1971.
3. NAZAROV G. I. and PUCHKOV A. A., The problem of the torsion of a hollow circular cylinder with variable shear moduli. *Prikl. Mat. Mekh.* **56**, 5, 871–873, 1992.
4. NAZAROV G. I. and PUCHKOV A. A., The torsion of an axisymmetrical body with mixed boundary conditions on the side surfaces. *Prikl. Mat. Mekh.* **36**, 6, 1094–1099, 1972.
5. NAZAROV G. I. and PUCHKOV A. A., The problem of the torsion of an inhomogeneous solid of rotation with variable shear moduli. *Prikl. Mekh.* **9**, 3, 32–37, 1973.
6. KRYLOV A. N., *Lectures on Approximate Calculations*. Gostekhizdat, Moscow, 1950.
7. BERGMAN S., *Integral Operators in the Theory of Linear Partial Differential Equations*. Mir, Moscow, 1964.
8. NAZAROV G. I., The accurate solution of the equations of gas dynamics. *Izv. Akad. Nauk SSSR. MZhG* 3, 113–120, 1968.

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